

Note on the self-duality of gauge fields in topologically nontrivial spacetime

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We show the derivation of the self-duality relation of abelian higher-form gauge field strength in the topologically nontrivial spacetime background. The so-called Pasti–Sorokin–Tonin action for the self-dual abelian gauge field assumes that the spacetime topology is trivial to derive the self-duality relation using a gauge transformation of the action. In this paper we find a new gauge transformation of the same theory and show that this new gauge transformation enables us to derive the self-duality relation even in the topologically nontrivial spacetime background.
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Subject Index B06, B25

1. Introduction

This paper considers the Lorentz covariant action, the so-called PST (Pasti–Sorokin–Tonin) action [1,2], for the self-dual field strength of the abelian higher-form gauge field on a general spacetime manifold. In particular, we focus on the derivation of the self-duality relation in the case where the spacetime has nontrivial cycles.

Recently [3] the PST action for the single M5-brane was applied to the check of the $\text{AdS}_7/\text{CFT}_6$ correspondence for the Wilson surface operators of 6D (2,0) A_n superconformal field theory (or its S^1 -reduction to 5D maximally supersymmetric Yang–Mills theory). This work obtained a nice correspondence, but there was a small subtlety. We explain this briefly.

In the gravity side, they computed the on-shell value of the PST action of a probe M5-brane that wraps S^3 in the $\text{AdS}_7 \times S^4$ background. Thus the worldvolume of the probe M5-brane has a nontrivial 3-cycle.

On the other hand, the recipe to derive the self-duality relation in the PST formalism is to find a general solution to the equation of motion and to gauge-transform the solution to produce the self-duality relation, the technical details of which will be given in the next section. For this recipe to work, we have to assume that the worldvolume is topologically trivial so that Poincaré’s lemma holds, otherwise the gauging-away to produce the self-duality relation does not work. In other words, the self-duality relation cannot be derived when the worldvolume has nontrivial cycles.

In summary, Ref. [3] obtained a nice correspondence using the PST action in the case where the self-duality relation cannot be derived, though the M5-brane worldvolume theory must have a self-dual gauge field. This situation may motivate us to think that the self-duality can be derived even when the worldvolume has nontrivial cycles.

In this paper, we claim that the self-duality relation can be derived even when the worldvolume has nontrivial cycles. To show this, we find a new gauge transformation (9), which may be regarded

as a modification of the gauge transformation that is used in the conventional derivation of the self-duality relation.¹ In the next section we will show the details of the new gauge transformation and the derivation of the self-duality relation.

2. PST action, gauge transformations, and the self-duality

PST action. We consider the Lorentz covariant action, the so-called PST action, for the self-dual field strength of the abelian $2n$ -form gauge field on a general manifold of dimensionality $D = 4n + 2$ ($n = 0, 1, 2$) with metric g . The properties of the action and the gauge transformations will be shown in an index-free manner. For definitions and notations, see the appendix.

Let M be a $(2n + 1)$ -form field strength. In this paper, we consider the case $M = dA$ for simplicity, where A is the $2n$ -form gauge potential.² The self-duality relation is $M = *M$. The PST action is given by [1,2,7]

$$S = \int \mathcal{L} = \int e_{\widehat{v}} \iota_v (M - *M) \wedge M, \quad (1)$$

where we have introduced a 1-form field \widehat{v} with an auxiliary 0-form field a , and its dual vector field v obtained by raising the index of \widehat{v} :

$$\widehat{v} := \frac{da}{\sqrt{(\partial a)^2}}, \quad v := \frac{\partial^a a}{\sqrt{(\partial a)^2}} \partial_a, \quad (\partial a)^2 := g^{ab} \partial_a a \partial_b a.$$

For the action in components, see the appendix. The action (1) has three gauge symmetries:

$$\delta_1 a = 0, \quad \delta_1 A = d\Lambda, \quad (2)$$

$$\delta_2 a = 0, \quad \delta_2 A = da \wedge \Phi, \quad (3)$$

$$\delta_3 a = \varphi, \quad \delta_3 A = \frac{\varphi}{\sqrt{(\partial a)^2}} \iota_v (M - *M), \quad (4)$$

where Λ and Φ are $(2n - 1)$ -forms and φ is a 0-form. The first gauge transformation δ_1 is the usual one. The second gauge symmetry δ_2 will be used to derive the self-duality condition from the equation of motion, as will be shown later. The third gauge symmetry δ_3 is used to eliminate the auxiliary field a . This is the origin of the modified Lorentz invariance of the actions [8,9] of self-dual gauge fields in which the Lorentz symmetry is not manifest.

The variation of the action (1) reads

$$\begin{aligned} \delta \mathcal{L} &= \frac{1}{\sqrt{(\partial a)^2}} d\delta a \wedge \widehat{v} \wedge [\iota_v (M - *M) \wedge \iota_v (M - *M)] - 2\delta M \wedge \widehat{v} \wedge \iota_v (M - *M) - M \wedge \delta M \\ &= -\delta a \cdot d \left[\frac{1}{\sqrt{(\partial a)^2}} \widehat{v} \wedge \iota_v (M - *M) \wedge \iota_v (M - *M) \right] + 2\delta A \wedge d[\widehat{v} \wedge \iota_v (M - *M)] \\ &\quad + (\text{total derivatives}), \end{aligned} \quad (5)$$

¹ A similar result was obtained in Ref. [4] for not manifestly Lorentz covariant action.

² When applied to the 10D IIB supergravity, the field strength is given by [5,6]

$$M := dA + (B \wedge H' - B' \wedge H)/2, \quad H := dB, \quad H' := dB',$$

where (B, B') is an $\text{SL}(2, \mathbb{Z})$ -doublet. Then, for the gauge invariance, which we will explain shortly, we need to add a topological term $A \wedge H \wedge H'$. This term itself had already been known before the PST action for the IIB SUGRA appeared [5,6]. Our new gauge invariance under (9) still holds in this case.

where we have used the formulae (A4) and the variations of \widehat{v} and ι_v on any n -form:

$$\begin{aligned}\delta\widehat{v} &= \frac{1}{\sqrt{(\partial a)^2}} \iota_v e_{\widehat{v}} d\delta a, \\ \delta(\iota_v \omega_n) &= \frac{1}{\sqrt{(\partial a)^2}} (-)^n * (\iota_v e_{\widehat{v}} d\delta a \wedge *^{-1} \omega_n) + \iota_v \delta \omega_n.\end{aligned}$$

It is very easy to see the invariance under the first gauge transformation δ_1 since $\delta_1 M = 0$. Let us consider the second gauge transformation δ_2 . The δa -term of (5) vanishes under δ_2 . We can show that the δM -term also vanishes since δM is a product of da and a $2n$ -form, and the δM -term contains \widehat{v} . Then the action is invariant under δ_2 . Finally, we can show the invariance under the third gauge transformation δ_3 using the relation

$$d\left(\frac{1}{\sqrt{(\partial a)^2}}\right) \wedge \widehat{v} = \frac{1}{\sqrt{(\partial a)^2}} d\widehat{v}.$$

Derivation of the self-duality relation. The equations of motion can be read from (5):

$$\delta a \quad : \quad d\left[\frac{1}{\sqrt{(\partial a)^2}} \widehat{v} \wedge \iota_v (M - *M) \wedge \iota_v (M - *M)\right] = 0, \quad (6)$$

$$\delta A \quad : \quad d[\widehat{v} \wedge \iota_v (M - *M)] = 0. \quad (7)$$

Note that (6) holds identically once (7) holds. Therefore the independent equation of motion is (7) only. The general solution to this takes the form

$$\widehat{v} \wedge \iota_v (M - *M) = da \wedge (d\eta + \omega), \quad (8)$$

where ω is closed but not exact, since, using (7),

$$\widehat{v} \wedge \iota_v (M - *M) = da \wedge (2n\text{-form}) \rightarrow d(2n\text{-form}) = 0 \rightarrow (2n\text{-form}) = d\eta + \omega.$$

As can be seen, the ω -term appears when the spacetime has nontrivial cycles. On the other hand, the δ_2 -variation of (8) reads $\delta_2[\widehat{v} \wedge \iota_v (M - *M)] = da \wedge d\Phi$. Then δ_2 allows us to eliminate the $d\eta$ -part of the solution (8) by setting $\Phi = -\eta$, while the ω -part cannot be eliminated by δ_2 . Therefore the self-duality relation cannot be derived when the spacetime background has nontrivial cycles.

New gauge transformation. Actually, we can show that the action (1) is invariant under

$$\delta'_2 a = 0, \quad \delta'_2 A = a\xi, \quad (9)$$

where ξ is closed but not necessarily an exact $2n$ -form. Then $\delta'_2 M = da \wedge \xi$. Since this has da , the variation of the action (5) vanishes. Thus this transformation can also be regarded as a gauge symmetry. Now the δ'_2 -variation of (8) reads $\delta'_2[\widehat{v} \wedge \iota_v (M - *M)] = -da \wedge \xi$. Therefore we can eliminate the ω -part of the solution. Since δ'_2 can also be used to eliminate the $d\eta$ -part of the solution (8) when

ξ is an exact form, the new transformation δ'_2 plays the same role as δ_2 . Thus we can adopt δ'_2 instead of δ_2 . Then the solution (8) can be made into

$$e_{\widehat{v}}\iota_v(M - *M) = 0.$$

Inserting $** = 1$ into this and using $e_{\widehat{v}}\iota_v* = *\iota_v e_{\widehat{v}}$, which can be derived from (A4), we obtain

$$\iota_v e_{\widehat{v}}(M - *M) = 0.$$

Then, summing the last two equations and using $\iota_v e_{\widehat{v}} + e_{\widehat{v}}\iota_v = 1$, we get the self-duality relation

$$M = *M.$$

Therefore we can obtain the self-duality relation even when the spacetime has nontrivial cycles.

3. Conclusion and discussion

In this paper, we have found a new gauge transformation that allows us to derive the self-duality relation even when the spacetime has nontrivial cycles. Our result can be applied to any actions and equations of motion for self-dual gauge fields that use a PST-like mechanism to derive the self-duality relations as the solution to the equations of motion, including DBI-like extensions such as in Refs. [1,2,7], actions with more than one auxiliary field such as in Ref. [10], actions that do not have manifest Lorentz covariance such as in Refs. [8,9,11], and the formalisms for non-abelian self-dual gauge fields such as in Ref. [12].

Our result will enlarge the range of applications of the PST action to physical systems in various spacetime backgrounds. In particular, it would be interesting to apply the formalism to the AdS/CFT correspondence with 10D IIB supergravity with self-dual Ramond–Ramond 5-form field strength, which we are now studying (H. Isono and T. Morita, manuscript in preparation).

It will be interesting to investigate the relation between the cohomology of the spacetime manifold and the structure of the physical states of systems of the self-dual gauge fields. The 2D case on a torus has been studied in Ref. [13], where the BRST analysis of the Floreanini–Jackiw action [14] on a torus was carried out and it was shown that the self-duality relation holds on the physical states. Our result can be regarded as the classical proof of this in more general spacetime backgrounds. However, the quantum analysis with the explicit BRST charge and the state space is much more important and strong. In particular, it would be interesting to see whether there are any nontrivial relations between the spacetime cohomology and the BRST cohomology for self-dual gauge field theories by the quantum BRST analysis.

Note added:. We found the new result in this paper long ago, and were preparing a paper (H. Isono and T. Morita, manuscript in preparation) that includes not only the result in this paper but also a new physical application of the PST action. Then we found Ref. [15], which proposes a gauge transformation that is similar to our new gauge transformation.

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Appendix A. Notations

Here we give the definitions of symbols used in the main part of the paper. We fix the dimensionality of the manifold to be D and the metric to be g . A k -form ω_k is related to its components by

$$\omega_k = \frac{1}{k!} \omega_{a_1 \dots a_k} dx^{a_1} \wedge \dots \wedge dx^{a_k}.$$

The Hodge star, which transforms k -form to $(D - k)$ -form, is defined so that

$$\eta_k \wedge * \theta_k = \frac{1}{k!} \eta_{a_1 \dots a_k} \theta^{a_1 \dots a_k} \sqrt{|g|} dx^1 \wedge \dots \wedge dx^D. \quad (\text{A1})$$

The Hodge star satisfies $**\omega_k = (-)^{k(D-k)} \text{sign}(g)\omega_k$.

The exterior product e_θ with a 1-form θ maps k -form to $(k + 1)$ -form and is defined by

$$e_\theta \omega := \theta \wedge \omega. \quad (\text{A2})$$

The interior product ι_v with a vector $v = v^a \partial_a$ maps k -form to $(k - 1)$ -form and is defined by

$$\iota_v \omega_k := \frac{1}{(k-1)!} v^a \omega_{aa_1 \dots a_{k-1}} dx^{a_1} \wedge \dots \wedge dx^{a_{k-1}}. \quad (\text{A3})$$

We will use the following useful formulae in deriving the variation of the action: on any k -form

$$\iota_v = (-)^{k+1} *^{-1} e_{\hat{v}} * = (-)^{D+k} * e_{\hat{v}} *^{-1}, \quad e_{\hat{v}} = (-)^k *^{-1} \iota_v * = (-)^{D+k+1} * \iota_v *^{-1}. \quad (\text{A4})$$

The action (1) in components reads

$$S = \int d^D x \frac{\sqrt{-g}}{(\partial a)^2} \partial^a a (M - *M)_{aa_1 \dots a_{2n}} (*M)^{a_1 \dots a_{2n} b} \partial_b a. \quad (\text{A5})$$

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